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The Existence of Polynomial Solution of the Nonlinear Dynamical Systems

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Abstract

The method to solve the nonlinear differential equation with variable coefficients is presented in this work. The method is based on the application of general Riccati equation, which is substituted into the considered nonlinear equation and produce polynomial equation. The polynomial order higher than four is reduced into the fourth order and solved by radicals. The solutions of the Riccati and polynomial are then combined performing the complete solution which is smooth for all time.

Keywords: Nonlinear ordinary differential equations, the Riccati equation, analytical solutions, dynamical systems

1. Introduction

Many of the more realistic modeling in the dynamical systems are based on nonlinear differential equations [1]. The application sometimes involve with variable coefficients [2]. It is well-known that the qualitative analysis of the nonlinear differential equations is sufficient to know the global behavior on the solutions [3,4]. However, the concepts will not be very useful until the explicit solutions are produced. The explicit solutions are capable to describe the detail features of the systems [5]. They may also help to extend the existence, uniqueness and regularity properties of the solutions which are obtained from qualitative analysis [6,7].

Therefore, method for generating analytical solutions of the nonlinear differential equations is important from both physical and mathematical point of views [8]. The case of control system with nonhomogenous physical property such as in vibration waves is of particular interest because it resulted in the model with variable coefficients. Since that specific problem attracts many mathematicians and

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physicists, the methods to obtain exact and approximate solutions are tackled systematically and some interesting results may be produced [6,9,10].

In this paper, the following class of equation will be discussed,

$$f_1\left(t,y,\frac{\partial y}{\partial t},....,\frac{\partial^m y}{\partial t^m}\right)\frac{\partial^n y}{\partial t^n}+f_2\left(t,y,\frac{\partial y}{\partial t},....,\frac{\partial^m y}{\partial t^m}\right)\frac{\partial^{n-1} y}{\partial t^{n-1}}+......+\sum_p f_p\left(t,y,\frac{\partial y}{\partial t},....,\frac{\partial^m y}{\partial t^m}\right)y^p=0 \quad (1)$$

which the method for obtaining exact solutions is conducted by using the Riccati equation. The procedure for obtaining the solutions is derived in a quite simple way which is based on the substitution into the considered equation. The resulting expression is then produced polynomial equation which then combined with the solution of the Riccati to form final solutions. The method that is explained in this work is new and the results may have significant contributions in the area of differential and integral equations.

2. Method of Generating Solutions

Let us consider the following nonlinear differential equations with variable coefficients,

$$F\left(t,y,\frac{\partial y}{\partial t},....,\frac{\partial^q y}{\partial t^q}\right)=0 \quad (2a)$$

The solution of (2a) is constructed from the following Riccati equation,

$$\frac{\partial y}{\partial t}=a_1y^2+a_2y+a_3 \quad (2b)$$

with, a_1,a_2 and a_3 are variable coefficients. Before substitute (2b) into (2a), the following relation is produced,

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial a_1}{\partial t}y^2+2a_1y\frac{\partial y}{\partial t}+\frac{\partial a_2}{\partial t}y+b_2\frac{\partial y}{\partial t}+\frac{\partial a_3}{\partial t}$$

$$\frac{\partial^2 y}{\partial t^2}=2a_1^2y^3+\left(\frac{\partial a_1}{\partial t}+3a_1a_2\right)y^2+\left(\frac{\partial a_2}{\partial t}+2a_1a_3+a_2^2\right)y+\frac{\partial a_3}{\partial t}+a_2a_3 \quad (2c)$$

$$\frac{\partial^3 y}{\partial t^3}=6a_1^3y^4+\left(12a_1^2a_2+6\frac{\partial a_1}{\partial t}a_1\right)y^3+\left(\frac{\partial^2 a_1}{\partial t^2}+8a_1^2a_3+5\frac{\partial a_1}{\partial t}a_2+7a_1a_2^2+4a_1\frac{\partial a_2}{\partial t}\right)y^2+$$
$$\left(\frac{\partial^2 a_2}{\partial t^2}+4\frac{\partial a_1}{\partial t}a_3+2a_1\frac{\partial a_3}{\partial t}+8a_1a_2a_3+3\frac{\partial a_2}{\partial t}a_2+a_2^3\right)y+\frac{\partial^2 a_3}{\partial t^2}+2\frac{\partial a_2}{\partial t}a_3+2a_1a_3^2+a_2^2a_3+a_2\frac{\partial a_3}{\partial t} \quad (2d)$$

....

.... (2e)

$$\frac{\partial^q y}{\partial t^q}=...........$$

The procedure described in (2b – d) will produce the following system,

$$\frac{\partial y}{\partial t}=a_1y^2+a_2y+a_3 \quad (2f)$$

$$y^r + a_{r+3}y^{r-1} + a_{r+2}y^{r-2} + ... + a_6y^2 + a_5y + a_4 = 0 \tag{2g}$$

where a_r are the coefficients from the substitution into (2a). Firstly, the Riccati equation is transformed as in the following,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right)\frac{\partial u}{\partial t} + a_1a_3u = 0 \tag{3a}$$

With $y = -\frac{1}{a_1}\frac{1}{u}\frac{\partial u}{\partial t}$.

Lemma1*: Consider equation (3a), let* $-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) = A_1 + \frac{1}{b_1}\frac{\partial b_1}{\partial t}$ *and* $a_1a_3 = A_2 + A_1\frac{1}{b_2}\frac{\partial b_2}{\partial t}$*, equation (3c) then has a closed-form exact solution which* b_1, b_2 *and* A_2 *are redefined. The solution will leave* A_1 *as an arbitrary function.*

Proof: Suppose, $-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) = A_1 + \frac{1}{b_1}\frac{\partial b_1}{\partial t}$ and $a_1a_3 = A_2 + A_1\frac{1}{b_2}\frac{\partial b_2}{\partial t}$, the above equation is transformed into,

$$\frac{1}{b_1}\frac{\partial}{\partial t}\left(b_1\frac{\partial u}{\partial t}\right) + \frac{A_1}{b_2}\frac{\partial(b_2u)}{\partial t} + A_2u = 0 \tag{3b}$$

Let, $A_2u = Z$, the equation can be rearranged as,

$$\frac{1}{b_1}\frac{\partial}{\partial t}\left[b_1\frac{\partial}{\partial t}\left(\frac{Z}{A_2}\right)\right] + \frac{A_1}{b_2}\frac{\partial}{\partial t}\left(b_2\frac{Z}{A_2}\right) + Z = 0 \text{ or}$$

$$\frac{\partial^2 Z}{\partial t^2} + \left[\frac{1}{b_1}\frac{\partial b_1}{\partial t} + A_1\frac{\partial}{\partial t}\left(\frac{1}{A_1}\right) + A_2\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) + A_1\right]\frac{\partial Z}{\partial t} +$$
$$\left[A_2\frac{\partial^2}{\partial t^2}\left(\frac{1}{A_2}\right) + A_2\frac{1}{b_1}\frac{\partial b_1}{\partial t}\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) + A_1\frac{1}{b_2}\frac{\partial b_2}{\partial t} + A_1A_2\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) + A_2\right]Z = 0 \tag{3c}$$

Let, $A_2\frac{\partial^2}{\partial t^2}\left(\frac{1}{A_2}\right) + A_2\frac{1}{b_1}\frac{\partial b_1}{\partial t}\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) + A_1\frac{1}{b_2}\frac{\partial b_2}{\partial t} + A_1A_2\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) + A_2 = 0$, the equation for A_2 is then,

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{A_2}\right) - \left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right)\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) + a_1a_3\left(\frac{1}{A_2}\right) = 0 \tag{3d}$$

Multiply by the function β and differentiate (3d) once to perform the non homogenous equation,

$$\frac{\partial^3}{\partial t^3}\left(\frac{1}{A_2}\right) + \left[-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{1}{\beta}\frac{\partial \beta}{\partial t}\right]\frac{\partial^2}{\partial t^2}\left(\frac{1}{A_2}\right) + \left[-\frac{\partial}{\partial t}\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + a_1a_3 - \left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right)\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right]\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right) +$$
$$\left(\frac{\partial a_1a_3}{\partial t} + a_1a_3\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right)\left(\frac{1}{A_2}\right) = 0 \tag{4a}$$

where β will be determined later. Suppose that,

$$-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+\frac{1}{\beta}\frac{\partial \beta}{\partial t}=A_3+\frac{b_{3x}}{b_3} \quad \text{and} \quad -\frac{\partial}{\partial t}\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+a_1a_3-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)\frac{1}{\beta}\frac{\partial \beta}{\partial t}=A_4+A_3\frac{b_{4x}}{b_4} \tag{4b}$$

The equation is then rewritten as,

$$\frac{\alpha}{b_3}\frac{\partial}{\partial t}\left[b_3\frac{\partial^2}{\partial t^2}\left(\frac{1}{A_2}\right)\right]+\alpha\frac{A_3}{b_4}\frac{\partial}{\partial t}\left[b_4\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right)\right]+\alpha A_4\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right)+\alpha\left[\left(\frac{\partial a_1a_3}{\partial t}+a_1a_3\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right)\right]\left(\frac{1}{A_2}\right)=0 \tag{4c}$$

If $\alpha_x A_4=\alpha\left(\frac{\partial a_1a_3}{\partial t}+a_1a_3\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right)$ then, $\alpha=C_1\exp\int_t\frac{1}{A_2}\left(\frac{\partial a_1a_3}{\partial t}+a_1a_3\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right)dt$, thus equation (4c) can be arranged as,

$$\frac{\alpha}{b_3}\frac{\partial}{\partial t}\left[\frac{b_3}{\alpha}\frac{\partial^2 H}{\partial t^2}+2b_3\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\frac{\partial H}{\partial t}+b_3\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)H\right]+\alpha\frac{A_3}{b_4}\frac{\partial}{\partial t}\left[\frac{b_4}{\alpha}\frac{\partial H}{\partial t}+b_4\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)H\right]+A_4\frac{\partial H}{\partial t}=0 \tag{5a}$$

with, $H=\alpha\left(\frac{1}{A_2}\right)$. The variables in (5a) is related as in the following,

$$\frac{\alpha}{b_3}\frac{\partial}{\partial t}\left[b_3\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\right]+\alpha\frac{A_3}{b_4}\frac{\partial}{\partial t}\left[b_4\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\right]=0 \text{ and}$$

$$\frac{\alpha}{b_3}\frac{\partial}{\partial t}\left(\frac{b_3}{\alpha}\right)+2\alpha\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)+A_3=-t\left\{\frac{\alpha}{b_3}\frac{\partial}{\partial t}\left[b_3\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\right]+\alpha\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)+\alpha\frac{A_3}{b_4}\frac{\partial}{\partial t}\left(\frac{b_4}{\alpha}\right)+\alpha A_3\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)+A_4\right\} \tag{5b}$$

The relation for b_3 is,

$$\frac{1}{b_3}\frac{\partial b_3}{\partial t}=A_3\frac{1}{b_4}\frac{\partial b_4}{\partial t}\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\right]^{-1}+A_3-\frac{\partial^3}{\partial t^3}\left(\frac{1}{\alpha}\right)\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\right]^{-1} \tag{5c}$$

Substitute into, $-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+\frac{1}{\beta}\frac{\partial \beta}{\partial t}=A_3+\frac{b_{3x}}{b_3}$ to give the expression for b_4 as,

$$\frac{b_{4r}}{b_4}=\frac{1}{A_3}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\left[\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\right]^{-1}\left\{\frac{\partial^3}{\partial t^3}\left(\frac{1}{\alpha}\right)\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\right]^{-1}-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right\}-2\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\left[\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\right]^{-1} \tag{5d}$$

Performing (5d) into, $-\frac{\partial}{\partial t}\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+a_1a_3-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)\frac{1}{\beta}\frac{\partial \beta}{\partial t}=A_4+A_3\frac{b_{4x}}{b_4}$, to produce A_3,

$$A_3=\frac{A_4}{2}-\frac{1}{2}\left[-\frac{\partial}{\partial t}\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+a_1a_3-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right]\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\right]^{-1}+ \\ \frac{1}{2}\left\{\frac{\partial^3}{\partial t^3}\left(\frac{1}{\alpha}\right)\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right)\right]^{-1}-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)+\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right\} \tag{5e}$$

The next step is to relate the function A_4, as $A_4=\frac{\partial a_1a_3}{\partial t}+a_1a_3\frac{1}{\beta}\frac{\partial \beta}{\partial t}$, which produce the solution of α as, $\alpha=C_1e^t$. Thus, by (5b) the equation for A_4 is,

$$A_4 = \frac{1}{t}\left[2t + \left(1 - \frac{b_{4r}}{b_4}\right)(t-1) - t\frac{b_{4r}}{b_4} - 1\right]A_3 + \frac{3}{t} + \frac{1}{t}(t-1) - 1 \text{ or}$$

$$A_4 = \frac{1}{t}(t+2)A_3 + \left\{\left[-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{1}{\beta}\frac{\partial \beta}{\partial t} - 1\right] - 2\right\}(t-1) + t\left\{-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{1}{\beta}\frac{\partial \beta}{\partial t} - 1\right\} + \frac{3}{t} + \frac{1}{t}(t-1) - 1$$

or

$$\left[1 - \frac{1}{2t}(t+2)\right]A_4 = \frac{1}{2}(t+2)\left[-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{1}{\beta}\frac{\partial \beta}{\partial t} - 1\right] + t\left[-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{1}{\beta}\frac{\partial \beta}{\partial t} - 1\right] + \frac{3}{t} + \frac{1}{t}(t-1) +$$
$$\frac{1}{2}(t+2)\left(\frac{\partial a_1 a_3}{\partial t} + a_1 a_3\frac{1}{\beta}\frac{\partial \beta}{\partial t}\right) + \left[-\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{1}{\beta}\frac{\partial \beta}{\partial t} - 3\right](t-1) - 1 \tag{6a}$$

which then solves β as in the following,

$$\beta = C_2 e^{\int_t \varphi dt} \tag{6b}$$

where,

$$\varphi = \frac{-(2.5t+1)\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t} + a_2\right) + \frac{2}{t} + (2 - 4.5t) - \left[1 - \frac{1}{2t}(t+2) - \frac{1}{2}(t+2)\right]\frac{\partial a_1 a_3}{\partial t}}{a_1 a_3\left[1 - \frac{1}{2t}(t+2) - \frac{1}{2}(t+2)\right] - (2.5t+1)} \tag{6c}$$

The equation for H becomes,

$$\frac{\partial^3 H}{\partial t^3} - t b_5 \frac{\partial^2 H}{\partial t^2} + b_5 \frac{\partial H}{\partial t} = 0 \tag{7a}$$

with,

$$b_5 = \frac{\alpha}{b_3}\frac{\partial}{\partial t}\left[b_3\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right)\right] + \alpha\frac{\partial^2}{\partial t^2}\left(\frac{1}{\alpha}\right) + \alpha\frac{A_3}{b_4}\frac{\partial}{\partial t}\left(\frac{b_4}{\alpha}\right) + \alpha A_3\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right) + A_4 = -\frac{1}{t}\left[\frac{\alpha}{b_3}\frac{\partial}{\partial t}\left(\frac{b_3}{\alpha}\right) + 2\alpha\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right) + A_3\right]$$

. Equation (7a) can be transformed into,

$$\frac{\partial K}{\partial t} + K^2 - t b_5 k + b_5 = 0$$

with, $\frac{\partial H}{\partial t} = e^{\int_t K dt}$. The above relation has $\frac{1}{t}$ as a particular solution, the general solution is governed by $K = \frac{1}{t} + \frac{1}{l}$, which resulted in,

$$-\frac{1}{l^2}\frac{\partial l}{\partial t} + \frac{1}{l^2} + \frac{2}{tl} - \frac{t b_5}{l} = 0 \text{ or } \frac{\partial l}{\partial t} = \left(\frac{2}{t} - t b_5\right)l + 1 \tag{7b}$$

The solution for $\frac{1}{A_2}$ is then,

$$\frac{1}{A_2}=\frac{1}{\alpha}H=\frac{1}{\alpha}\left(\int_t e^{\int_t Kdt}dt+C_4\right)=\frac{1}{\alpha}\left(\int_t e^{\int_t \frac{1}{t}+\frac{1}{l}dt}dt+C_4\right)=$$

$$C_1^{-1}e^{-t}\left\{\int_t e^{\int_t \frac{1}{t}+\exp\left(-\int_t \frac{2}{t}-tb_5 dt\right)\left[\int_t \exp\left(-\int_t \frac{2}{t}-tb_5 dt\right)dt+C_3\right]^{-1}dt}dt+C_4\right\}=C_1^{-1}e^{-t}\left[\int_t t\left(\int_t \frac{1}{t^2}e^{\int_t tb_5 dt}dt+C_3\right)dt+C_4\right] \tag{7c}$$

The equation (3c) becomes,

$$\frac{\partial^2 Z}{\partial t^2}+\left[\frac{1}{b_1}\frac{\partial b_1}{\partial t}+A_1\frac{\partial}{\partial t}\left(\frac{1}{A_1}\right)+A_2\frac{\partial}{\partial t}\left(\frac{1}{A_2}\right)+A_1\right]\frac{\partial Z}{\partial t}=0 \tag{8a}$$

The solution for the Riccati equation is then,

$$y=-\frac{1}{a_1}\frac{1}{u}\frac{\partial u}{\partial t}=-\frac{1}{a_1}\frac{A_2}{Z}\frac{\partial}{\partial t}\left(\frac{Z}{A_2}\right)=$$

$$-\frac{1}{a_1}A_2\left[C_1\int_t A_1A_2e^{\int_t\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)dt}dt+C_2\right]^{-1}\frac{\partial}{\partial t}\left\{\frac{1}{A_2}\left[C_1\int_t A_1A_2e^{\int_t\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)dt}dt+C_2\right]\right\} \tag{8b}$$

where A_2 is determined by (7c).

Thus, the procedure leaves A_1 as an undefined variable, this proves lemma 1

Since the solution of the polynomial by radical is limited to the fourth order, the reduction of polynomial order should be performed. The interested reader will find the method of reduction in [11].

Theorem 2: *Consider the solution of the equation (3c) as described by (8b). By combining with the root of polynomial,* $y=\phi(t)$, *the resulting expressions thus complete the solution of the system defined byRiccati and polynomial equations.*

Proof: Let $y=\phi$ is the polynomial solution, the combination with (8b) will determined A_1 as,

$$A_1=C_3\frac{1}{A_2}e^{-\int_t\left(\frac{1}{a_1}\frac{\partial a_1}{\partial t}+a_2\right)dt}\frac{\partial}{\partial t}\left(A_2e^{-\int_t a_1\phi dt}\right) \tag{8c}$$

which then proves theorem 2.

3. Solution Properties

Now we are at step to answer and proof the questions of existence and uniqueness of smooth solution. Since the coefficients of the Riccati equation are arbitrary, they can become powerful objects to justify the properties under general initial-boundary conditions.

3.1. Uniqueness Property

Let us consider the second order ODE which the solution and initial condition are related as, $Y = y_1 - y_2$ and $Y(0) = y_1(0) - y_2(0) = 0$. Substituting the solution pairs into (8b) will then produce a unique solution for y since it is from linear ODE. As for the polynomial solution, under the proper selection of coefficients will also produce a unique solution.

3.2. Existence and Regularity Properties

Apart from uniqueness, the existence and regularity properties depend on the chosen function of the variable coefficients of Riccati equations, a_i. Also the proper selection of coefficients will produce $\phi < \infty$ for all time.

4. Conclusions

The method for the generating solution of the nonlinear differential equation is proposed in this article. The main strategy is to substitute the Riccati equation into the considered equation. The resulted polynomial is then solved by radicals and combined with the solution of the Riccati equation. It is shown that the method can obtain the solutions of arbitrary coefficients and arbitrary order in closed-form. The solution is exist and smooth for all time. We plan to conduct the applications in our future works.

References

[1] Barrio RA and Varea C, Non-Linear Systems, *Physica A 372*, 2006, pp. 210 – 223.

[2] Sadri S, Raveshi MR and Amiri S, Efficiency Analysis of Straight Fin with Variable Heat Transfer Coefficient and Thermal Conductivity, *Journal of Mechanical Science and Technology 26 (4),* 2012, pp. 1283 – 1290.

[3] Doherty MFand Ollio JM, Chaos in Deterministic Systems: Strange Attractors, Turbulence and Applications in Chemical Engineering, *Chemical Engineering Science*, Vol. 43, No. 2, 1988, pp. 139 – 183.

[4] Calogero F, Gomez-Ullate D, Santini PM and Sommacal M, Towards a Theory of Chaos Explained as Travel on Riemann Surfaces, *J. Phy. A: Math. Theor. 42*, 2009.

[5] Abdel-Gawad HI, On the Behavior of Solutions of a Class of Nonlinear Partial Differential Equations, *Journal of Statistical Physics*, 1999, Vol. 97, Nos. 1/2.

[6] Ramos JI, Analytical and Approximate Solutions to Autonomous , Nonlinear, Third-Order Ordynary Differential Equations, *Nonlinear Analysis: Real World Applications 11*, 2010, pp. 1613 – 1636.

[7] Heywood JG, Nagata W and Xie W, A Numerically Based Existence Theorem for the Navier-Stokes Equations, *Journal of Mathematical Fluid Mechanics 1*, 1999, pp. 5 – 23.

[8] Galaktonov VA and Svirshchevskii AR, *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*, Taylor & Francis Group, Boca Raton, 2007.

[9] Bougoffa L, On the Exact Solutions for Initial Value Problems of Second Order Differential Equations, *Applied Mathematics Letters 22*, 2009, pp. 1248 – 1251.

[10] Adomian G, *Solving Frontier Problem of Physics: The Decomposition Method*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.

[11] Nugroho G, Application of First Order Polynomial Differential Equation ofr Generating Analytical Solutions to the Three-Dimensional Incompressible Navier-Stokes Equations, *European Journal of Mathematical Sciences*, Vo. 2, N0. 1, 2013, pp. 17 – 40.